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COMPLEMENTED COPIES OF ℓ_2 IN SPACES OF INTEGRAL OPERATORS

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Abstract. It is shown that, if E and F are Banach spaces containing complemented copies of ℓ_1, then the space of integral operators $\mathcal{I}(E, F^*) \equiv (E \otimes_\epsilon F)^*$ contains a complemented copy of ℓ_2. This answers a question of Félix Cabello and Ricardo García.

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By E and F we denote Banach spaces, E^* is the dual of E, and B_E is the closed unit ball of E; $\mathcal{L}(E, F)$ is the space of all (bounded linear) operators from E into F, endowed with the uniform norm. By $E \otimes_\pi F$ (respectively, $E \otimes_\epsilon F$) we represent the projective (respectively, injective) tensor product of E and F. The notation $\mathcal{I}(E, F)$ stands for the space of all integral operators from E into F, endowed with the integral norm, while $\mathcal{L}_{\mathrm{I}}(^2E)$ denotes the space of all integral bilinear forms on E. By the symbol $E \equiv F$ we mean that E and F are isometrically isomorphic. The set of all natural numbers is denoted by $\mathbb{N}$, and $\mathbb{K}$ stands for the scalar field.

Recall that we have

$$(c_0 \otimes_\pi c_0)^{**} \equiv (\ell_1 \otimes_\epsilon \ell_1)^* \equiv \mathcal{I}(\ell_1, \ell_\infty) \equiv \mathcal{L}_{\mathrm{I}}(^2\ell_1);$$

(see [**6**, Definition VIII.2.6 and Corollary VIII.2.12] and [**8**, p. 787]).

It was proved in [**7**, Theorem 10] that the above spaces do not have the Dunford-Pettis property. In [**2**], F. Cabello and R. García asked if they contain a complemented reflexive subspace. We show that they do contain a complemented copy of ℓ_2.

THEOREM 1. *The space $\mathcal{I}(\ell_1, \ell_\infty)$ contains a complemented copy of ℓ_2.*

Proof. The proof relies mainly on two facts: the existence of a surjective operator $\ell_\infty \to \ell_2$, and the fact that the formal inclusion $\ell_1 \to \ell_2$ is absolutely summing [**5**, Theorem 1.13].

Let $q : \ell_\infty \to \ell_2$ be a surjective operator [**5**, Corollary 4.16]. Then there are $C > 0$ and a sequence $(\phi^n)_{n=1}^\infty \subset \ell_\infty$ with $\|\phi^n\| \le C$ such that $q(\phi^n) = e^n$, for all $n \in \mathbb{N}$, where $e^n = (0, \dots, 0, 1, 0, \dots)$ with 1 in the n-th position.

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Let $T : \mathcal{I}(\ell_1, \ell_\infty) \to \ell_2$ be given by

$$T(A) := (\langle e^i, qAe^i \rangle)_{i=1}^\infty \qquad \text{for } A \in \mathcal{I}(\ell_1, \ell_\infty).$$

By the proof of [**7**, Theorem 10], T is a well defined operator.

Let $J : \ell_2 \to \mathcal{L}(\ell_1, \ell_\infty)$ be given by

$$J(\alpha)(x) := \sum_{j=1}^\infty \alpha_j x_j \phi^j, \qquad \text{for } \alpha = (\alpha_j)_{j=1}^\infty \in \ell_2, \quad x = (x_j)_{j=1}^\infty \in \ell_1.$$

We have

$$\|J(\alpha)(x)\| = \left\| \sum_{j=1}^\infty \alpha_j x_j \phi^j \right\| \leq \sum_{j=1}^\infty |\alpha_j| \cdot |x_j| \cdot \|\phi^j\| \leq C\|\alpha\|_\infty \cdot \|x\|_1 \leq C\|\alpha\|_2 \cdot \|x\|_1,$$

and so $J(\alpha) \in \mathcal{L}(\ell_1, \ell_\infty)$. Moreover,

$$\|J(\alpha)\| = \sup \left\{ \|J(\alpha)(x)\| : x \in B_{\ell_1} \right\} \leq C \cdot \|\alpha\|_2,$$

and J is continuous.

We now show that $J(\alpha) \in \mathcal{I}(\ell_1, \ell_\infty)$; equivalently, the bilinear form

$$\Psi_\alpha : \ell_1 \times \ell_1 \longrightarrow \mathbb{K},$$

given by

$$\Psi_\alpha(x, y) := \langle J(\alpha)(x), y \rangle = \left\langle \sum_{j=1}^\infty \alpha_j x_j \phi^j, y \right\rangle,$$

is integral [**6**, Corollary VIII.2.12]. By [**6**, Definition VIII.2.6], we have to show that its linearization $\overline{\Psi_\alpha}$ belongs to $(\ell_1 \otimes_\epsilon \ell_1)^*$.

Let $B_\alpha : \ell_2 \times \ell_1 \to \mathbb{K}$ be given by

$$B_\alpha(x, y) := \left\langle \sum_{j=1}^\infty \alpha_j x_j \phi^j, y \right\rangle \qquad \text{for } x = (x_j)_{j=1}^\infty \in \ell_2, \quad y \in \ell_1.$$

Since

$$\left\| \sum_{j=1}^\infty \alpha_j x_j \phi^j \right\|_\infty \leq \sup_{j \in \mathbb{N}} \|\phi^j\| \cdot \sum_{j=1}^\infty |\alpha_j x_j| \leq C \cdot \|x\|_2 \cdot \|\alpha\|_2,$$

we have that B_α is continuous with $\|B_\alpha\| \leq C \cdot \|\alpha\|_2$.

Denoting by I_{ℓ_1} the identity map on ℓ_1, since the natural inclusion $I_2 : \ell_1 \to \ell_2$ is absolutely summing, the operator

$$I_2 \otimes I_{\ell_1} : \ell_1 \otimes_\epsilon \ell_1 \longrightarrow \ell_2 \otimes_\pi \ell_1$$

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is continuous [**4**, Proposition 11.1]. We have

$$\begin{aligned}\overline{B_\alpha}(I_2 \otimes I_{\ell_1})\left(\sum_{i=1}^{n} x^i \otimes y^i\right) &= \overline{B_\alpha}\left(\sum_{i=1}^{n} x^i \otimes y^i\right)\\ &= \sum_{i=1}^{n} B_\alpha(x^i, y^i)\\ &= \sum_{i=1}^{n}\left\langle \sum_{j=1}^{\infty} \alpha_j x_j^i \phi^j, y^i\right\rangle\\ &= \sum_{i=1}^{n} \Psi_\alpha(x^i, y^i)\\ &= \overline{\Psi_\alpha}\left(\sum_{i=1}^{n} x^i \otimes y^i\right),\end{aligned}$$

for $y^i \in \ell_1$ and $x^i = (x_j^i)_{j=1}^{\infty} \in \ell_1$ $(1 \leq i \leq n)$. Hence

$$\overline{B_\alpha}(I_2 \otimes I_{\ell_1}) = \overline{\Psi_\alpha},$$

and $\overline{\Psi_\alpha} \in (\ell_1 \otimes_\epsilon \ell_1)^*$. Therefore, $J(\alpha) \in \mathcal{I}(\ell_1, \ell_\infty)$.

Moreover,

$$\|J(\alpha)\|_{\mathrm{I}} = \|\Psi_\alpha\|_{\mathrm{I}} = \|\overline{\Psi_\alpha}\| = \left\|\overline{B_\alpha}(I_2 \otimes I_{\ell_1})\right\| \leq \|\overline{B_\alpha}\| = \|B_\alpha\| \leq C \cdot \|\alpha\|_2,$$

and so $J : \ell_2 \to \mathcal{I}(\ell_1, \ell_\infty)$ is continuous.

Now, for $\alpha \in \ell_2$, we have

$$q(J(\alpha)(e^i)) = q(\alpha_i \phi^i) = \alpha_i e^i,$$

so that

$$T(J(\alpha)) = (\langle e^i, q(J(\alpha)(e^i))\rangle)_{i=1}^{\infty} = (\langle e^i, \alpha_i e^i\rangle)_{i=1}^{\infty} = (\alpha_i)_{i=1}^{\infty} = \alpha,$$

and $TJ = I_{\ell_2}$. Therefore, JT is a projection. □

The following Corollary is now clear.

COROLLARY 2. *Suppose that E and F contain complemented copies of ℓ_1. Then the space $(E \otimes_\epsilon F)^*$ contains a complemented copy of ℓ_2. Moreover, the space $(E \otimes_\epsilon F)^*$ does not have the Dunford-Pettis property.*

The fact that $(\ell_1 \otimes_\epsilon \ell_1)^*$ fails to have the Dunford-Pettis property was established in [**7**, Theorem 10].

REMARK 3. (a) We do not know if there are spaces E and F whose duals have the Dunford-Pettis property, fulfilling the hypotheses of [**7**, Theorem 10] and which do not satisfy the conditions of Corollary 2; that is, such that at least one of them contains no complemented copy of ℓ_1.

(b) The author is grateful to the referee for pointing out that the space $\mathcal{P}_{\mathrm{I}}(^2\ell_1)$ of integral 2-homogeneous scalar-valued polynomials on ℓ_1 is isomorphic to $\mathcal{L}_{\mathrm{I}}(^2E)$

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[**1**, Corollary 4.4]. Therefore, $\mathcal{P}_{\mathrm{I}}({}^2\ell_1)$ also contains a complemented subspace isomorphic to ℓ_2.

(c) While this paper was submitted, Ignacio Villanueva kindly sent to the author a draft of [**3**], where Theorem 1 is proved by different techniques.

REFERENCES

1. J. M. Ansemil and K. Floret, The symmetric tensor product of a direct sum of locally convex spaces, *Studia Math.* **129** (1998), 285–295.

2. F. Cabello Sánchez and R. García, The bidual of a tensor product of Banach spaces, *Rev. Mat. Iberoamericana*, to appear.

3. F. Cabello, D. Pérez-García, and I. Villanueva, Complemented copies of Hilbert spaces in the tensor products of Banach spaces, article in preparation.

4. A. Defant and K. Floret, *Tensor norms and operator ideals,* Math. Studies **176** (North-Holland, 1993).

5. J. Diestel, H. Jarchow, and A. Tonge, *Absolutely summing operators*, Cambridge Stud. Adv. Math. **43** (Cambridge University Press, 1995).

6. J. Diestel and J. J. Uhl, Jr., *Vector measures*, Math. Surveys Monographs **15** (American Mathematical Society, Providence RI, 1977).

7. M. González and J. M. Gutiérrez, The Dunford-Pettis property on tensor products, *Math. Proc. Cambridge Philos. Soc.* **131** (2001), 185–192.

8. R. A. Ryan, The Dunford-Pettis property and projective tensor products, *Bull. Polish Acad. Sci. Math.* **35** (1987), 785–792.